Exactly Measurable Concurrence of Mixed States

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We, for the first time, show that bipartite concurrence for rank 2 mixed states of qubits is written by an observable which can be exactly and directly measurable in experiment by local projective measurements, provided that four copies of the composite quantum system are available. In addition, for a tripartite quantum pure state of qubits, the 3-tangle is also shown to be measurable only by projective measurements on the reduced density matrices of a pair of qubits conditioned on four copies of the state.

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I. INTRODUCTION

Quantum entanglement has been realized to be a useful physical resource for quantum information processing. The quantification of entanglement has attracted increasing interests in recent years [1]. However, entanglement *per se* is not an observable in strict quantum mechanical sense. That is to say, up till now, no directly measurable observable corresponds to entanglement of a given arbitrary quantum state, owing to the unphysical quantum operations in usual entanglement measure [2], for example, the complex conjugation of concurrence [3] and the partial transpose of negativity [4,5]. A general approach to characterize entanglement in experiment is quantum state tomography, by which one needs to first reconstruct the density matrix by measuring a complete set of observables [6-8] and then turn to the mathematical problem of evaluating some entanglement measure. But this approach has been implemented successfully in experiment which is only suitable for small quantum systems because the number of measured observables grows rapidly with the dimension of the system. Entanglement witnesses are also effective for the detection of entanglement [9] if priori knowledge on the states is available, which depends on the detected states. Quite recently, some approaches have been reported for the determination of entanglement in experiment [2,10-12,14-16]. The most remarkable are the new formulation of concurrence [13] in terms of copies of the state which led to the first direct experimental evaluation of entanglement [12] and some analogous contributions [14] to multipartite concurrence. These are only restrictive to pure states. However, in reality quantum states are never really pure, hence it is very necessary to consider how to determine the entanglement of a mixed state in experiment.

In this paper, we study the exact and measurable entanglement of mixed states. Recently concurrence in terms of copies of states was generalized to bipartite [15] and multipartite mixed states [16] such that concurrence of mixed states can be obtained by effective experimental estimations. However, both the two estimations in fact only provide observable lower bounds of concurrence with purity dependent tightness for mixed states instead of the exact concurrence. One has to accept the fact that the exactly measurable entanglement for a general mixed state is still a challenge. Here we only take the first step to focus on concurrence for rank 2 bipartite mixed states of qubits. The exact value instead of lower bound is given by a Hermite operator (observable) and shown to be directly measurable in experiment based on local projective measurements, provided that four copies of the tripartite quantum pure state are available. It is quite interesting that, for a given tripartite quantum pure state of qubits, its 3-way entanglement measure i.e., 3-tangle can be directly measured only by the local projective measurements on the reduced density matrices of any two qubits without the requirement. of performing any operation on the third qubit.

The paper is organized as follows. In Sec. II, we show that the exact concurrence is directly and locally measurable in experiment, and that the exact 3-tangle for a tripartite quantum pure state of qubits is also measurable conditioned on fourfold copy, respectively. The conclusion is drawn in Sec. III.

II. MEASURABLE CONCURRENCE

We first show that the concurrence of a rank-2 mixed state is exactly measurable. The concurrence of a given rank 2 mixed state of a pair of qubits ρ_{AB} is given [3,17] by

$$C\left(\rho_{AB}\right)=\left|\lambda_1-\lambda_2\right|, \tag{1}$$

where λ_i are the square roots of the eigenvalues of $\rho_{AB}\tilde{\rho}_{AB}$ in decreasing order with $\tilde{\rho}_{AB}=\left(\sigma_y\otimes\sigma_y\right)\rho^*_{AB}\left(\sigma_y\otimes\sigma_y\right)$. After a simple algebra, one has

$$C^2\left(\rho_{AB}\right)=Tr\left(\rho_{AB}\tilde{\rho}_{AB}\right)-\tau, \tag{2}$$

where

$$\tau=\sqrt{2\left\{\left[Tr\left(\rho_{AB}\tilde{\rho}_{AB}\right)\right]^2-Tr\left[\left(\rho_{AB}\tilde{\rho}_{AB}\right)^2\right]\right\}} \tag{3}$$

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is half of the 3-tangle [18,19] of the tripartite quantum pure state of qubits such that ρ_{AB} is the reduced density matrix. It is easy find that

$$Tr\left(\rho_{AB}\tilde{\rho}_{AB}\right)=\sqrt{Tr[(\rho_{AB}\otimes\rho_{AB})\mathcal{B}]}, \tag{4}$$

with $\mathcal{B}=4P_-^{A_1A_2}\otimes P_-^{B_1B_2}$, where

$$P_-^{(i_m i_n)}=\frac{1}{\sqrt{2}}\left(\left|0\right\rangle_{i_m}\left|1\right\rangle_{i_n}-\left|1\right\rangle_{i_m}\left|0\right\rangle_{i_n}\right), \tag{5}$$

denoting the projector onto the anti-symmetric subspace $\mathcal{H}_{i_m}\wedge\mathcal{H}_{i_n}$ of $\mathcal{H}_{i_m}\otimes\mathcal{H}_{i_n}$ where $i=A,B$ corresponds to the subsystems, and $m,n=1,2,3,4$ marks the different copies of ρ_{AB}. $Tr\left(\rho_{AB}\tilde{\rho}_{AB}\right)$ has been written in the form of the expectation value of the self-adjoint operator $\mathcal{B}$ which is measured by local projective measurements on two copies of ρ_{AB}, hence it is measurable. The remaining are to show that $Tr\left[\left(\rho_{AB}\tilde{\rho}_{AB}\right)^2\right]$ is also directly measurable in experiment.

Consider a decomposition of

$$\rho_{AB}=\sum_i\left|\psi_i\right\rangle_{AB}\left\langle\psi_i\right|=\sum_i p_i\left|\varphi_i\right\rangle_{AB}\left\langle\varphi_i\right|, \tag{6}$$

with $\sum\limits_i p_i=1$ and $\left|\psi_i\right\rangle_{AB}=\sqrt{p_i}\left|\varphi_i\right\rangle_{AB}$, $Tr\left[\left(\rho_{AB}\tilde{\rho}_{AB}\right)^2\right]$ can be given by

$$\begin{aligned}
& Tr\left[\left(\rho_{AB}\tilde{\rho}_{AB}\right)^2\right]\\
=\ & Tr\sum_{ijkl}\left|\psi_i\right\rangle^1\left\langle\psi_i\right|\Sigma\left|\psi_j^*\right\rangle^2\left\langle\psi_j^*\right|\Sigma\left|\psi_k\right\rangle^3\left\langle\psi_k\right|\Sigma\left|\psi_l^*\right\rangle^4\left\langle\psi_l^*\right|\Sigma\\
=\ & \sum_{ijkl}\left\langle\psi_i\right|^1\left\langle\psi_j\right|^2\mathbf{P}\left|\psi_j\right\rangle^2\left|\psi_k\right\rangle^3\left\langle\psi_k\right|^3\left\langle\psi_l\right|^4\mathbf{P}\left|\psi_l\right\rangle^4\left|\psi_i\right\rangle^1\\
=\ & \sum_{ijkl}\left\langle\psi_i\right|^1\left\langle\psi_j\right|^2\left\langle\psi_k\right|^3\left\langle\psi_l\right|^4\left(\mathbf{P}\otimes\mathbf{P}\right)\left|\psi_j\right\rangle^2\left|\psi_k\right\rangle^3\left|\psi_l\right\rangle^4\left|\psi_i\right\rangle^1\\
=\ & Tr\left[\otimes_{i=1}^4\rho_{AB}^i\left(\mathbf{P}\otimes\mathbf{P}\right)SWAP\right]=Tr\left[\otimes_{i=1}^4\rho_{AB}^i\mathcal{A}\right],
\end{aligned} \tag{7}$$

with

$$\mathcal{A}=\frac{\left(\mathbf{P}\otimes\mathbf{P}\right)SWAP+SWAP^{\dagger}\left(\mathbf{P}\otimes\mathbf{P}\right)}{2}, \tag{8}$$

$$\mathbf{P}=P_-\otimes P_- \tag{9}$$

where P_- without superscripts and subscripts means a general projector onto the anti-symmetric subspace of two qubits, $\Sigma=\sigma_y\otimes\sigma_y$, $SWAP$ is defined as

$$\left|\psi_j\right\rangle^2\left|\psi_k\right\rangle^3\left|\psi_l\right\rangle^4\left|\psi_i\right\rangle^1=SWAP\left|\psi_i\right\rangle^1\left|\psi_j\right\rangle^2\left|\psi_k\right\rangle^3\left|\psi_l\right\rangle^4,$$

and all $\left|\psi\right\rangle$ denote bipartite pure state with subscirpts AB omitted. The superscripts of $\left|\psi\right\rangle$ in eq. (7) mark the different copies of ρ_{AB}. The last "=" in eq. (7) follows from the fact that $Tr\left[\left(\rho_{AB}\tilde{\rho}_{AB}\right)^2\right]$ should not be changed if the different copies are exchanged. It happened that $\mathcal{A}$ can always be written by

$$\mathcal{A}=\frac{1}{2}\left(M_A\otimes M_B-N_A\otimes N_B\right),$$

where $M_A=M_B$ and $N_A=N_B$, M_i and N_i denote the observables (Hermite operators) of the four copies of the ith subsystem. In principle, one can always obtain the projectors of M_i and N_i by their eigenvalue decompositions and perform corresponding local projective measurements on the *fourfold* subsystems. We have found that both M_i and N_i are rank 2, hence $\mathcal{A}$ can be directly measured by 8 projectors. That is to say, the exact concurrence can be directly measured by *10* local projective measurements (plus $\mathcal{B}$).

In fact one can reduce the number of projective measurements further. After a simple derivation, one can find that M_i can always be written by

$$M_i=\frac{\sqrt{2}}{2}\left(P_-^{i_1i_2}\otimes P_-^{i_3i_4}-P_-^{i_1i_3}\otimes P_-^{i_2i_4}-P_-^{i_1i_4}\otimes P_-^{i_2i_3}\right). \tag{10}$$

Thus $M_A\otimes M_B$ corresponds to *9* groups of projective measurements (*9* observables). Consider the invariance of the exchange of different copies, one can find that $M_A\otimes M_B$ can be reduced to *2* groups of projective measurements as

$$\begin{aligned}
M_A\otimes M_B\ =\ & \frac{1}{2}\Big(3P_-^{A_1A_2}\otimes P_-^{A_3A_4}\otimes P_-^{B_1B_2}\otimes P_-^{B_3B_4}\\
& -2P_-^{A_1A_2}\otimes P_-^{A_3A_4}\otimes P_-^{B_1B_3}\otimes P_-^{B_2B_4}\Big)
\end{aligned} \tag{11}$$

$Tr\left(\rho_{AB}\tilde{\rho}_{AB}\right)$ is also included in the two measurements. N_i can be explicitly by

$$N_i=\sqrt{3}\left(\left|\Psi\right\rangle\left\langle\Psi\right|-\left|\bar{\Psi}\right\rangle\left\langle\bar{\Psi}\right|\right), \tag{12}$$

where

$$\begin{aligned}
\left|\Psi\right\rangle\ =\ & \frac{1}{\sqrt{6}}\Big(\left|0011\right\rangle+e^{\frac{2\pi i}{3}}\left|0101\right\rangle+e^{-\frac{2\pi i}{3}}\left|0110\right\rangle\\
& +e^{-\frac{2\pi i}{3}}\left|1001\right\rangle+e^{\frac{2\pi i}{3}}\left|1010\right\rangle+\left|1100\right\rangle\Big)
\end{aligned} \tag{13}$$

is the four-qubit Dicke state [21] with two excitations if neglecting the relative phases and $\left|\bar{\Psi}\right\rangle$ is the conjugate $\left|\Psi\right\rangle$. However, so far we can not find a proper decomposition in terms of two-qubit projectors for N_i (Strictly speaking, we can not find such decompositions that reduce the number of observables). That is to say, the projectors of N_i have to be performed on a four-qubit space as a whole. In this sense, only *6* groups of local projective measurements are enough for concurrence.

More interestingly, from eq. (3) one can find that τ (3-tangle) for a tripartite pure state of qubits can be directly measured only by local projective measurements on the subsystems of two qubits. Consider a tripartite quantum pure state of qubits $\left|\Psi\right\rangle_{ABC}$ with the reduced density ρ_{AB}, then ρ_{AB} is obviously rank-two. Based on

ref. [18,19], 3-tangle of $|\Psi\rangle_{ABC}$ can be given by eq. (3). The above procedure implies eq. (3) is measurable, i.e. 3-tangle is directly measurable if four copies of ρ_{AB} are available. Different from our previous work [20] that requires projective measurements on all subsystems, only projective measurements on two subsystems (reduced density matrix of two qubits) are enough, even though so far projection measurements on four-qubit space are required.

We have expressed concurrence of rank-two mixed states by Hermite operators which can be decomposed into local projectors. This shows in principle that the exact concurrence can be directly and locally measured in experiment. However, as mentioned above, it requires projective measurements on the whole space of four qubits, which means the interactions of four qubits. Take a photon experiment as an example, it might need the interference of four photons. In the current experiment, together with four copy of a state, it might be very difficult to realize, hence present scheme might introduce no advantage in current experiment. But we can safely say that only two qubit interaction can not be enough for the really applicable quantum computer. In this sense, the current work should be a valuable contribution for the further research. In addition, it is still an open problem whether the observables given here are derived from an optimal decomposition such that the number of observables are minimal and whether there exist some other decompositions of N_i or $\mathcal{A}$ which lead to a direct contribution to the current experiments. This is our forthcoming efforts.

III. CONCLUSION AND DISCUSSION

We have shown that the exact bipartite concurrence for rank 2 mixed states of qubits instead of a lower bound is directly and locally measurable in experiment, provided that four copies of the states of interests are available. In particular, it is very interesting that 3-tangle for a tripartite quantum pure state of qubits has been shown to be measurable experimentally if only the *fourfold* copy of bipartite reduced density matrix is available, which is different from our previous work [20]. Furthermore, because a state has to be prepared repeatedly in order to obtain reliable measurement statistics in any experiment [2], a fourfold copy of a state should be feasible in principle, which implies the observation of mixed-state entanglement may be feasible. However, practical experiments may lead to four different "copies" of a state and influence the fidelity, which requires that one has to perform necessary error analysis based on different experimental realization [2].

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